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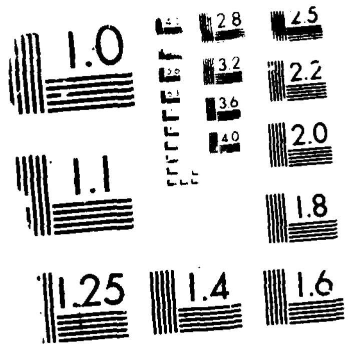

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Report 024713-2-T

OUTPUT AIMING CONTROL

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OUTPUT AIMING CONTROL

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ABSTRACT

The problem of aiming control, introduced in [1], is extended to systems with outputs. Necessary and sufficient conditions for output residence time controllability in linear SISO systems with small, additive noise are derived. Controller design techniques are developed and applied to aircraft and robotics control problems. The approach is based on an extension of the asymptotic first passage time theory to output processes.

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1. INTRODUCTION

Given a controlled dynamical system with states $x(t) \in I\!R^n$, control $u(t) \in I\!R^m$, output $y(t) \in I\!R^p$ and disturbances $\xi(t) \in I\!R^r$, we define the *output aiming process specifications* as a pair $\{\Psi,\tau\}$, where $\Psi \subset I\!R^p$ is the domain to which the outputs $y(t)$ should be confined and τ is the period of the confinement, i.e., $y(t) \in \Psi, \forall\ t \in [t_o, t_o + \tau], t_o \in I\!R_+$.

For a given pair $\{\Psi,\tau\}$, the problem of *output aiming control* is formulated as the problem of choosing a feedback control law so as to force the average duration of $y(t)$ in Ψ to be larger than τ, in spite of the disturbances $\xi(t)$ that are acting on the system.

An analogous problem, concerning the aiming of the states $x(t)$, rather than the outputs $y(t)$, has been formulated and analyzed in [1]. In particular, it has been shown that the existence of a desired aiming controller for linear systems with small additive, white noise depends on the relationship between the column spaces of the control and noise matrices. If the former includes the latter, any precision of aiming is possible (strong residence time controllability). If this inclusion does not occur, the achievable precision of aiming is bounded (weak residence time controllability).

The purpose of the present paper is to analyze the fundamental capabilities and limitations of *output* aiming control for linear systems with additive perturbations. Specifically, we show that a linear SISO system which is weakly residence time controllable in states may, in fact, be strongly residence time controllable in the output if its nonminimum phase zeros satisfy a certain property. As far as the design of aiming controllers is concerned, we show that H^2-minimization of the closed loop transfer

function leads to maximization of the accuracy of output aiming whereas unweighted H^∞-minimization leads to its minimization.

The structure of the paper is as follows: in Section 2 the notion of an output residence time is introduced; in Section 3 output residence time controllability is defined and analyzed; in Section 4 output aiming controller design techniques are given; in Section 5 examples are considered and Section 6 is devoted to conclusions. The proofs are given in the Appendix.

2. OUTPUT RESIDENCE TIME

Consider a linear stochastic system

$$\begin{aligned} dx &= Axdt + \varepsilon C dw \\ y &= Dx \end{aligned} \tag{2.1}$$

where $x \in I\!R^n$, $y \in I\!R^p$, $w(t)$ is a standard r-dimensional Brownian motion and $0<\varepsilon \ll 1$ is a parameter.

Let $\Psi \subset I\!R^p$ be an open bounded domain containing the origin and whose boundary $\partial\Psi$ is smooth. Assume that $x_0=x(0) \in \Omega_0 \stackrel{\Delta}{=} \{x \in I\!R^n \mid y=Dx \in \Psi\}$ and denote as $y(t,x_0)$ the output $y(t)$ defined by (2.1) with the initial condition x_0. Introduce the first passage time of the output $y(t,x_0)$ from Ψ as follows:

$$\tau^\varepsilon(x_0)=\inf\{t \geq 0: y(t,x_0) \in \partial\Psi \mid y(t_0,x_0) \in \Psi\} \tag{2.2}$$

and its mean

$$\bar{\tau}^\varepsilon(x_0) = E[\tau^\varepsilon(x_0) \mid x_0]\,. \tag{2.3}$$

The calculation of $\overline{\tau}^{\varepsilon}(x_0)$ is, in general, a difficult task. To alleviate this difficulty, asymptotic approximations with respect to small ε can be used. For the special case $y(t) \equiv x(t)$ these approximations have been extensively discussed in the literature (see, e.g., [1] - [3] and references therein). An extension to the more general case of $y(t) = Dx(t)$ is given below (it is assumed, without loss of generality, that rank $D = p$).

Theorem 2.1: Assume

(i) A is Hurwitz,

(ii) (A, C) is completely disturbable, i.e.,

rank $[C \;\; AC \;\; \cdots \;\; A^{n-1}C] = n$.

Then uniformly for all x_0 *belonging to compact subsets* of $\Omega = \{x_0 \in \mathbb{R}^n \mid De^{At}x_0 \in \Psi, t \geq 0\}$ we have

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln \overline{\tau}^{\varepsilon}(x_0) = \hat{\mu}(\Psi) \quad , \tag{2.4}$$

where

$$\hat{\mu}(\Psi) = \min_{y \in \partial\Psi} \frac{1}{2} y^T N y \quad , \tag{2.5a}$$

$$N = (DXD^T)^{-1} \tag{2.5b}$$

and X is the positive definite solution of

$$AX + XA^T + CC^T = 0 \quad . \tag{2.5c}$$

Proof: See the Appendix.

The constant $\hat{\mu}(\Psi)$ is referred to as the *logarithmic residence time* in Ψ. The properties of this constant, as stated in Theorem 2.1, constitute the mathematical founda-

tion for the analysis in Sections 3 and 4.

If y and w are scalars, the logarithmic residence time can be expressed in a more traditional form. Indeed, since in this case Ψ is an interval, say, $\Psi = (-a, b)$, $a, b > 0$, and $G_n(s) \stackrel{\Delta}{=} D(sI - A)^{-1} C$ is a scalar, from (2.5) it follows that

$$\hat{\mu}(\Psi) = \frac{1}{2} (\min(a,b))^2 N,$$

$$N = \left[\int_0^\infty De^{At} CC^T e^{A^T t} D^T dt \right]^{-1} = \left[\int_0^\infty (De^{At} C)^2 dt \right]^{-1}$$

$$= \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |D(j\omega I - A)^{-1} C|^2 dw \right]^{-1} = \|G_n\|_2^{-2} .$$

Therefore,

$$\hat{\mu}(\Psi) = \frac{(\min(a,b))^2}{2\|G_n\|_2^2} . \tag{2.6}$$

3. OUTPUT RESIDENCE TIME CONTROLLABILITY

Consider now a controlled linear stochastic system

$$\begin{aligned} dx &= (Ax + Bu)dt + \varepsilon C dw \; , \; u \in \mathbb{R}^m \\ y &= Dx \; . \end{aligned} \tag{3.1}$$

Definition: (i) The output $y(t)$ of system (3.1) is said to be *weakly residence time controllable* (y-wrt controllable) if for any bounded domain $\Psi \subset \mathbb{R}^p$ with 0 in its interior, there exists a control $u = Kx$ such that $\hat{\mu}(\Psi, K) > 0$.

(ii) $y(t)$ is said to be *strongly residence time controllable* (y-srt controllable) if for any bounded $\Psi \subset \mathbb{R}^p$ ($0 \in \Psi$) and $\mu > 0$ there exists $u = Kx$ such that $\hat{\mu}(\Psi, K) \geq \mu$.

Here, $\hat{\mu}(\Psi,K)$ is the *precision of aiming* in Ψ,

$$\hat{\mu}(\Psi,K) = \lim_{\varepsilon\to 0} \varepsilon^2 \ln \bar{\tau}^{\varepsilon}(x_0,K) \quad , \tag{3.2}$$

where $\bar{\tau}^{\varepsilon}(x_0,K)$ is the mean first passage time in Ψ of the closed loop system

$$\begin{aligned} dx &= (A + BK)x dt + \varepsilon C dw \\ y &= Dx \quad , \end{aligned} \tag{3.3}$$

with initial conditions $x_0 = x(0) \in \Omega_K = \{x \in I\!R^n \mid De^{(A+BK)t}x \in \Psi,\ t \geq 0\}$.

In [1] we have discussed wrt- and srt-controllability of the state vector $x(t)$, i.e., x-wrt and x-srt controllability. The main result is the following:

Theorem 3.1: Assume (3.1) has no modes that are both uncontrollable and undisturbable. Then

(i) (3.1) is x-wrt controllable if and only if (A, B) is stabilizable.

(ii) (3.1) is x-srt controllable if and only if (A, B) is stabilizable and $\mathrm{Im}C \subseteq \mathrm{Im}B$.

Remark 3.1: The assumption that (3.1) has no modes that are both uncontrollable and undisturbable, i.e. $(A,[B\ \ C])$ is a controllable pair, is made to rule out some mathematical degeneracies. Methods for relaxing this assumption are discussed in [1].

The following theorem characterizes the class of y-wrt controllable systems.

Theorem 3.2: Assume that $(A,[B\ \ C])$ is controllable. Then (3.1) is y-wrt controllable if (A,B) is stabilizable. If (D,A) is detectable then stabilizability of (A,B) is also a necessary condition for y-wrt controllability.

Proof: See the Appendix.

Thus, y-*wrt controllability* of a detectable output is equivalent to x-wrt controllability. Finding conditions which ensure y-srt controllability is a more difficult problem. To simplify the situation, in the remainder of the paper we assume that y, u and w are scalars and address the following problems:

Problem 1: Under which conditions is a stabilizable system (3.1) y-srt controllable?

Problem 2: What is the fundamental bound on the achievable precision of output aiming of an output which is not y-srt controllable?

Problem 3: How to design a controller which results in a desired precision of output aiming?

We give the solutions to Problems 1 and 2 in this Section and to Problem 3 in Section 4. The assumption of Theorem 3.2 will be assumed to hold in the remainder of the paper.

The following theorem solves Problem 1.

Theorem 3.3: Assume (A, B) is stabilizable. Then (3.1) is y-srt controllable if and only if all the nonminimum phase zeros of $G_s(s) \triangleq D(sI - A)^{-1}B$ are also zeros of $G_n(s) = D(sI - A)^{-1}C$.

Proof: See the Appendix

Remark 3.2: If the open right half plane zeros of $G_s(s)$ and $G_n(s)$ are disjoint, then (3.1) is y-srt controllable if and only if $G_s(s)$ is minimum phase. On the other

hand, if (3.1) is x-srt controllable, i.e. Im $C \subseteq$ Im B, then $G_n(s)$ is proportional to $G_s(s)$ and, therefore, by Theorem 3.3 any output is y-srt controllable.

Let K denote the class of all stabilizing controllers for systems (3.1), i.e.

$$\mathrm{K} = \{K \mid A + BK \text{ is Hurwitz}\}$$

and define the *maximum achievable precision of aiming* in Ψ by

$$\mu^*(\Psi) = \sup_{K \in \mathrm{K}} \hat{\mu}(\Psi, K) \ . \tag{3.4}$$

Obviously, $\mu^*(\Psi) = \infty$ for an srt-controllable output. Let $z_1, \cdots, z_l$ be the open right half plane (rhp) zeros of $G_s(s)$.

Theorem 3.4: Assume (A, B) is stabilizable. Then $\mu^*(\Psi)$ is given by

$$\mu^*(\Psi) = \frac{(\min(a, b))^2}{2\|G_0\|_2^2} \ , \tag{3.5}$$

$$G_0(s) = \frac{q(s)}{\prod_{i=1}^{l} (\bar{z}_i + s)} \ , \tag{3.6}$$

where $\bar{z}_i$ is the complex conjugate of z_i and $q(s)$ is the unique polynomial of degree less than l determined by the interpolation constraints: At each rph zero z of $G_s(s)$ of multiplicity m, $G_0(s)$ satisfies

$$\frac{d^k}{ds^k} G_0(s)\Big|_{s=z} = \frac{d^k}{ds^k} G_n(s)\Big|_{s=z}, \ k=0, \cdots, m-1 \ . \tag{3.7}$$

Proof: See the Appendix

Remark 3.3: If (3.1) is y-srt controllable then the interpolation constraints (3.7) become

$$\frac{d^k}{ds^k} G_0(s) \Big|_{s=z} = 0 \ , k = 0, \ldots, m-1 \ .$$

The unique polynomial $q(s)$ of degree less than l which satisfies these constraints is the zero polynomial. Therefore, $G_0(s) = 0$ as should be expected.

Remark 3.4: The function $G_0(s)$ defined by (3.6) is the rational function of minimum H^2-norm which satisfies the interpolation constraints (3.7) [4], [5]. Thus, the problem (3.4) is equivalent to the problem

$$\min\{ \|G\|_2 : G(s) \in H^2, G(s) \text{ rational}\}$$

subject to the constraints (3.7). On the other hand, the H^∞-optimal function which satisfies the interpolation constraints (3.7) is an all pass function, i.e., constant in magnitude on the $j\omega$-axis. Thus, an unweighted H^∞-optimal controller leads to zero precision of aiming.

The formula (3.5) for $\mu^*(\Psi)$ simplifies considerably when the rhp zeros of $G_s(s)$ are distinct. Define an $l \times l$ matrix Z by $z_{ij} = (z_i + \bar{z}_j)^{-1}, 1 \le i, j \le l$, and an $l \times 1$ column vector g by $g_j = G_n(z_j), j = 1, \cdots, l$.

Theorem 3.5: Assume that the rhp zeroes of $G_s(s)$ are all distinct. Then

$$\mu^*(\Psi) = \frac{(\min(a,b))^2}{2g^H Z^{-1} g} \tag{3.8}$$

where g^H is the Hermitian transpose of g, i.e. $g^H = \bar{g}^T$.

Proof: See the Appendix

4. OUTPUT AIMING CONTROLLER DESIGN

In this section we give a method for selecting a controller which results in any admissible precision of aiming $\hat{\mu} < \mu^*(\Psi)$.

Recall that $\hat{\mu}(\Psi, K)$ is given by

$$\hat{\mu}(\Psi, K) = \frac{1}{2}(\min(a, b))^2 N(K) \quad . \tag{4.1}$$

Thus, the maximization problem (3.4) is equivalent to the minimization problem

$$\inf_{K \in \mathrm{K}} DX(K)D^T \quad . \tag{4.2}$$

Since the equation which $X(K)$ satisfies is linear in K, it is easy to see that the infimum (4.2) is not attained at any $K \in \mathrm{K}$. Thus, $\mu^*(\Psi)$ is not attained for any $K \in \mathrm{K}$. We now construct a sequence of controllers whose precisions of aiming converge to $\mu^*(\Psi)$. Let K_0 be a stabilizing feedback for (3.1) and define a regularized "cost"

$$J_\gamma(K) = DX(K_0+K)D^T + \gamma KX(K_o+K)K^T, \gamma>0 \; . \tag{4.3}$$

Obviously,

$$\hat{\mu}(\Psi, K_0 + K) \geq \frac{(\min(a, b))^2}{2J_\gamma(K)} \quad , K \in \mathrm{K} \quad . \tag{4.4}$$

It is well known from the theory of optimal control that $J_\gamma(K)$ is minimized by

$$K^\gamma = -\frac{1}{\gamma} B^T Q_\gamma \tag{4.5}$$

where Q_γ is the positive semi-definite solution of

$$(A + BK_0)^T Q_\gamma + Q_\gamma(A + BK_0) + D^T D - \frac{1}{\gamma} Q_\gamma BB^T Q_\gamma = 0 \quad . \tag{4.6}$$

The following Theorem can be proved using the results of [6].

Theorem 4.1: Assume $(A\ B)$ is stabilizable. Then

(i) $\lim_{\gamma\to 0} \hat{\mu}(\Psi, K_0 + K^{\gamma}) = \mu^*(\Psi)$;

(ii) $\hat{\mu}(\Psi, K_0 + K^{\gamma})$ is nondecreasing as $\gamma \to 0$.

Thus, Theorem 4.1 provides an iterative method for finding a controller which results in any admissible precision of aiming. Indeed, for any desired $\hat{\mu} < \mu^*(\Psi)$ one simply iteratively finds a $\gamma_1 > 0$ such that

$$DX(K_o + K^{\gamma_1})D^T \leq \frac{(\min(a,b))^2}{2\hat{\mu}} .$$

Furthermore, the iterative process is simplified by the monotonicity of $\hat{\mu}(\Psi, K_o + K^{\gamma})$ in γ.

5. EXAMPLES

Example 5.1: In [1] we considered the problem of designing a roll attitude regulator for a missle disturbed by random torques. The system is described by

$$\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta \\ \omega \\ \phi \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \varepsilon \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \dot{w} \qquad (5.1)$$

where δ is the aileron deflection, ω is the roll angular velocity, ϕ is the roll angle, u is a command signal to aileron actuators and $\dot{w}$ is white noise.

Clearly $\mathrm{Im}C \not\subseteq \mathrm{Im}B$ so (5.1) is not x-srt controllable. However, as was indicated in [7] the main objective of the control is to maintain the roll angle ϕ within desired limits during the operation of the missle. Thus we choose an output $y = \phi$ and

investigate the residence time controllability of this output. A simple calculation shows that the transfer function from u to y is

$$G_s(s) = \frac{10}{s^2(s+1)} \quad .$$

Thus, since $G_s(s)$ has no zeros the output y is srt-controllable and any precision of aiming is achievable.

Example 5.2: Consider the problem of controlling the tip position of a flexible robot arm using control torques applied at the robot arms hub [8]. A finite dimensional approximate model for a robot arm which is flexible in the horizontal plane but not in the vertical plane or in torsion was described in [8]. The model is described by the following set of equations

$$\dot{x}_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i\omega_i \end{bmatrix} x_i + \begin{bmatrix} 0 \\ \frac{1}{I_T} \frac{d\phi_i}{dz}(0) \end{bmatrix} u + \begin{bmatrix} \phi_i(L) \\ 0 \end{bmatrix} \xi \tag{5.3}$$

$$y_i = [\phi_i(L) \ \ 0] x_i \ , \ i = 0, 1, \ldots, n \ ,$$

where L is the length of the arm, ζ_i, ω_i and $\phi_i(z)$ ($z \in [0,L]$) are the damping coefficient, pinned-free frequency and modal gain, respectively, of the i-th mode of oscillation, I_T is the total moment of inertia, u is the control torque, ξ is a random torque acting on the tip, and $y = \sum_{i=0}^{n} y_i$ is the tip position.

Assume that it is desired to maintain the tip position within the bounds $-a \le y \le b$ during a specified time-interval T, and assume that the disturbance ξ can be modelled as a small white noise $\varepsilon\dot{w}$.

The system transfer function for (5.3) is

$$G_s(s) = \frac{1}{I_T} \sum_{i=0}^{n} \frac{\phi_i(L)\frac{d\phi_i(0)}{dz}}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} \tag{5.4}$$

and the noise transfer function is

$$G_n(s) = \sum_{i=0}^{n} \frac{\phi_i^2(L)(s+2\zeta_i\omega_i)}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} \tag{5.5}$$

It was indicated in [8] that taking $n = 3$ gives a good approximate model and the values of the constants $\zeta_i, \omega_i, \phi_i(L), d\phi_i(0)/dz$ and I_T were determined experimentally. The resulting system has three right half plane zeroes at $z_1 = 12.04$ and $z_{2,3} = 21.5 \pm j\ 25.3$. It is easily checked that $G_n(z_1) \neq 0$. Thus system (5.3) is not y-srt controllable. However, it is controllable and, thus, y-wrt controllable and the maximal achievable precision of aiming can be obtained from (3.8) to be

$$\begin{aligned} \mu(\Psi) &= \frac{(\min(a,b))^2}{2g^H Z^{-1} g} \\ &= \frac{(\min(a,b))^2}{2(0.046)} = 10.87(\min(a,b))^2. \end{aligned} \tag{5.6}$$

Here,

$$Z^{-1} = \begin{bmatrix} 0.167 & -0.014-j0.136 & -0.049+j0.136 \\ -0.049+j0.136 & 74.178 & -31.150+j38.462 \\ -0.049-j0.136 & -31.150-j38.462 & 74.178 \end{bmatrix}$$

and

$$g = \begin{bmatrix} 0.181 \\ 0.092 - j0.057 \\ 0.092 + j0.057 \end{bmatrix}$$

Thus, any specified time-interval $[0,T]$ has to satisfy the bound

$$\ln T \le \frac{10.87(\min(a,b))^2}{\varepsilon^2} \quad . \tag{5.7}$$

6. CONCLUSIONS

In this paper we have discussed the problem of aiming control, introduced in [1], for linear Itô-type systems with outputs. Among the results, the following are of importance:

(a) y-wrt controllability of any detectable output is equivalent to x-wrt controllability;

(b) A stabilizable SISO system is y-srt controllable if and only if $H(s) = G_s(s)G_n^{-1}(s)$ is minimum phase;

(c) A system which is not x-srt controllable can be y-srt controllable.

The y-srt controllability was discussed only for systems with scalar outputs. The results can be extended to multivariable systems with fewer outputs than inputs in a quite straightforward way. However, all proofs become considerably more involved. For systems with fewer inputs that outputs the problem is much more difficult and results are only known for the special case $y = x$.

APPENDIX

Proof of Theorem 2.1: First note that it follows from the definition of Ω_0 that

$$\inf\{t \geq 0 \mid y(t) \in \partial\Psi\} = \inf\{t \geq 0 \mid x(t) \in \partial\Omega_0\} \; . \tag{A.1}$$

We will show that the first exit of $x(t)$ from Ω takes place on the boundary of Ω_0. Thus, for $x_0 = x(0) \in \Omega$ the first exit of $y(t)$ from Ψ is equivalent to the first exit of $x(t)$ from Ω. Then we will show that the logarithmic residence time in Ω (and, thus, in Ψ) is given by (2.5).

Let X be the positive definite solution of (2.5c) and let $M = X^{-1}$. Consider the sets

$$S_m = \{x \in I\!R^n \mid \frac{1}{2} x^T M x < m\} \; . \tag{A.2}$$

For sufficiently small m, $S_m \subset \Omega_0$. Furthermore, $S_m \rightarrow I\!R^n$ as $m \rightarrow \infty$. Thus, there exists a largest $\hat{m}$ such that $S_{\hat{m}} \subset \Omega_0$. Note that $S_{\hat{m}}$ is an invariant set of the system $\dot{x} = Ax$. Furthermore, Ω is the largest invariant set of $\dot{x} = Ax$ contained in Ω_0. Therefore, $S_{\hat{m}} \subset \Omega$. Next note that the mean first passage time

$$\overline{\tau}_1^{\varepsilon}(x_0) = E[\inf\{t \geq 0 \mid x(t) \in \partial\Omega\} \mid x_0 \in \Omega]$$

of

$$dx = Axdt + \varepsilon C dw$$

from Ω satisfies, uniformly for x_0 belonging to compact subsets of Ω [1],

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \ln \overline{\tau}_1^{\varepsilon}(x_0) = \hat{\phi}(\Omega) = \min_{x \in \partial\Omega} \frac{1}{2} x^T M x \; . \tag{A.3}$$

Since $\hat{m}$ is the largest m such that $S_m \subset \Omega_0$ it follows that $\frac{1}{2} x^T M x = \hat{m}$ at some

points $x \in \partial\Omega_0$. Furthermore, since $S_{\hat{m}} \subset \Omega \subset \Omega_0$ it follows that these points also belong to the boundary of Ω. It follows from (A.3) that $\hat{\phi}(\Omega) \leq \hat{m}$. If $\hat{\phi}(\Omega) < \hat{m}$ then the points where the minimum (A.3) is attained belong to the interior of $S_{\hat{m}}$ which contradicts $S_{\hat{m}} \subset \Omega$. Therefore, $\hat{\phi}(\Omega) = \hat{m}$ and the points where the minimum (A.3) is attained belong to the boundary of Ω_0. Also, as $\varepsilon \to 0$ the points of exit of $x(t)$ from Ω converge to the points where the minimum (A.3) is attained [9]. Therefore, it follows that $x(t)$ exits Ω at a boundary point of Ω_0 in the limit $\varepsilon \to 0$. Thus from (A.1) and the above discussion we have

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln \bar{\tau}^{\varepsilon}(x_0) = \hat{\phi}(\Omega) \tag{A.4}$$

uniformly for x_0 belonging to compact subsets of Ω.

Next we show that $\hat{\phi}(\Omega) = \hat{\mu}(\Psi)$. Let $x = T\bar{x}$ be a nonsingular change of coordinates that maps (2.1) into the form

$$d\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = T^{-1}AT\bar{x}dt + \varepsilon T^{-1}Cdw$$
$$= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} dt + \varepsilon \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix} dw$$

$$y = DT\bar{x} = \bar{D}\bar{x} = [I \;\; 0] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

(such a T always exists because rank $D = p$). Under this change of coordinates M is mapped into $\bar{M} = T^T MT$ and the domains Ω_0 and Ω become

$$\bar{\Omega}_0 = \{\bar{x} \in I\!R^n \mid \bar{x}_1 \in \Psi\}$$

and

$$\tilde{\Omega} = \{\tilde{x}_0 \in \mathbb{R}^n \mid y = \tilde{D} e^{\tilde{A}t} \tilde{x}_0 \in \Psi,\ t \geq 0\} \ .$$

The logarithmic residence time in $\tilde{\Omega}$(and $\tilde{\Omega}_0$) is now given by

$$\begin{aligned} \hat{\phi}(\tilde{\Omega}) &= \min_{\tilde{x} \in \partial\tilde{\Omega}_0} \frac{1}{2}\ \tilde{x}^T\ \tilde{M}\ \tilde{x} \\ &= \min_{\substack{\tilde{x}_1 \in \partial\Psi \\ \tilde{x}_2 \text{ free}}} \frac{1}{2}\ [\tilde{x}_1^T\ \tilde{x}_2^T] \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^T & \tilde{M}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \ . \end{aligned} \tag{A.6}$$

We minimize first with respect to the unconstrained variables $\tilde{x}_2$ giving

$$\tilde{x}_2 = -\tilde{M}_{22}^{-1}\ \tilde{M}_{12}^T\ \tilde{x}_1 \ . \tag{A.7}$$

Substituting (A.7) into (A.6) and rearranging gives

$$\hat{\phi}(\tilde{\Omega}) = \min_{\tilde{x}_1 \in \partial\Psi} \frac{1}{2}\ \tilde{x}_1[\tilde{M}_{11} - \tilde{M}_{12}\ \tilde{M}_{22}^{-1}\ \tilde{M}_{12}^T]\tilde{x}_1 \ . \tag{A.8}$$

The matrix $\tilde{M}_{11} - \tilde{M}_{12}\ \tilde{M}_{22}^{-1}\ \tilde{M}_{12}^T$ is exactly $\tilde{X}_{11}^{-1}$ where

$$\tilde{M}^{-1} = \tilde{X} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^T & \tilde{X}_{22} \end{bmatrix} \ .$$

Therefore,

$$\hat{\phi}(\tilde{\Omega}) = \min_{\tilde{x}_1 \in \partial\Psi} \frac{1}{2}\ \tilde{x}_1^T\ \tilde{X}_{11}^{-1}\ \tilde{x}_1 \ . \tag{A.9}$$

However, $\tilde{X}_{11} = \tilde{D}\tilde{X}\tilde{D}^T$ and therefore

$$\hat{\phi}(\tilde{\Omega}) = \min_{\tilde{x}_1 \in \partial\Psi} \frac{1}{2}\ \tilde{x}_1^T\ (\tilde{D}\tilde{X}\tilde{D}^T)^{-1}\tilde{x}_1 \ . \tag{A.10}$$

Finally, substituting back the original coordinates gives $\tilde{D}\ \tilde{X}\ \tilde{D}^T$

$= DTT^{-1}X(T^T)^{-1}\ T^T D^T = DXD^T$. Thus

$$\hat{\phi}(\tilde{\Omega}) = \hat{\mu}(\Psi) = \frac{1}{2} \min_{y \in \partial\Psi} \frac{1}{2}\ y^T N y \ . \tag{A.11}$$

Q.E.D.

Proof of Theorem 3.2: The sufficiency part of the theorem follows directly from Theorem 3.1 of [1].

To prove the necessity note that y-wrt controllability implies that there exists a control $u = Kx$ such that $DX(K)D^T > 0$. Assume, without loss of generality, that the closed loop system has the Kalman canonic form, i.e.,

$$A + BK = \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{33} & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} \end{bmatrix} \tag{A.12}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}, D = [D_1 \; 0 \; D_3 \; 0] .$$

The subsystem $\left(\begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right)$ is controllable and the detectability of (D, A) implies that $\tilde{A}_{44}$ is Hurwitz. If we can show that $\tilde{A}_{33}$ is Hurwitz the proof is complete.

Note that

$$DX(K)D^T = [D_1 D_3] \begin{bmatrix} X_{11} & X_{13} \\ X_{13}^T & X_{33} \end{bmatrix} \begin{bmatrix} D_1^T \\ D_3^T \end{bmatrix} = \hat{D}\hat{X}\hat{D}^T \tag{A.13}$$

where X_{ij}, $1 \le i, j \le 4$, is a decomposition of $X(K)$ compatible with (A.12). Also, $\hat{X}$ satisfies the Liapunov equation

$$\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{C}\hat{C}^T = 0 \tag{A.14}$$

where

$$\hat{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{13} \\ 0 & \tilde{A}_{33} \end{bmatrix}, \hat{C} = \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} .$$

The pair $(\hat{D}, \hat{A})$ is observable. Therefore, by [10, Corollary 1] all eigenvalues of $\hat{A}$ in Re $s \geq 0$ are undisturbable. However, since $[A, [B\ C])$ is controllable, we can assume that $(A + BK, C)$ is a disturbable pair (otherwise an arbitrarily small change in K, say δK, will render $(A + B(K + \delta K), C)$ disturbable [11]). Therefore, $(\hat{A}, \hat{C})$ is a disturbable pair and, thus, $\hat{A}$ is Hurwitz. **Q.E.D.**

Proof of Theorem 3.3: We know from (2.6) and Theorem 4.1 that

$$\inf_{K \in \mathbf{K}} DX(K)D^T = \lim_{\gamma \to 0} \| G_\gamma \|_2^2 , \tag{A.15}$$

where

$$G_\gamma(s) = G_n(s, \overline{K}_\gamma) = D(sI - A - B\overline{K}_\gamma)^{-1}C, \ \overline{K}_\gamma = K_0 + K^\gamma . \tag{A.16}$$

Assume first that $\sup_{K \in \mathbf{K}} \hat{\mu}(K) = \infty$ (i.e., (3.1) is y-srt controllable). Thus, by (A.15) and (4.1), $\lim_{\gamma \to 0} \| G_\gamma \| = 0$ or equivalently $\lim_{\gamma \to 0} G_\gamma(s) = 0$. Note that $G_\gamma(s)$ can be rewritten as

$$\begin{aligned} G_\gamma(s) &= D(sI - A - B\overline{K}_\gamma)^{-1}C \\ &= D[I - (sI - A)^{-1}B\overline{K}_\gamma]^{-1}(sI - A)^{-1}C \\ &= D[I + (sI - A - B\overline{K}_\gamma)^{-1}B\overline{K}_\gamma](sI - A)^{-1}C \\ &= D(sI - A)^{-1}C + D(sI - A - B\overline{K}_\gamma)^{-1}B\overline{K}_\gamma(sI - A)^{-1}C \\ &= G_n(s) + G_s(s, \overline{K}_\gamma)T_\gamma(s) . \end{aligned} \tag{A.17}$$

Let

$$G_n(s) = \frac{a(s)}{d(s)} \tag{A.18a}$$

$$G_s(s,\bar{K}_\gamma) = \frac{n(s)}{d_\gamma(s)} \tag{A.18b}$$

$$T_\gamma(s) = \frac{m_\gamma(s)}{d(s)} \tag{A.18c}$$

and note that $d_\gamma(s) = \det(sI - A - B\bar{K}_\gamma)$ is Hurwitz for all $\gamma > 0$ and $m_\gamma(s) = \bar{K}_\gamma \operatorname{adj}(sI - A)C$ is a polynomial of degree less than n. Write $n(s) = n_l(s)n_r(s)$ where $n_l(s)$ has zeroes only in Re $s \le 0$ and $n_r(s)$ has zeroes in Re $s > 0$ only. It can be shown that $\sqrt{\gamma}\, d_\gamma(s) \to n_l(s)n_r(-s)$ as $\gamma \to 0$ [12] and $\sqrt{\gamma}\, \bar{K}_\gamma \to \bar{K}$ as $\gamma \to 0$ [6]. Thus,

$$0 = G_0(s) = \lim_{\gamma \to 0} G_\gamma(s) = G_n(s) + \frac{n(s)m_0(s)}{n_l(s)n_r(-s)d(s)} \tag{A.19}$$

where $m_0(s) = \bar{K} \operatorname{adj}(sI - A)C$. Equation (A.19) can be rewritten as

$$G_0(s) = G_n(s) + G_s(s)\, T_0(s) = 0 \tag{A.20}$$

where $T_0(s) = m_0(s)/n_l(s)n_r(-s)$. Evaluating $G_0(s)$ at a right half plane zero of $G_s(s)$ gives $G_0(z) = G_n(z) = 0$ (because $n_l(z)\, n_r(-z) \ne 0$). Similarily, if z is a rhp zero of $G_\gamma(s)$ of multiplicity m, then it follows from (A.20) that the first $m-1$ derivatives of $G_n(s)$ are zero at z, i.e., $G_n(s)$ has also zero of at least multiplicity m at z. This completes the proof of the necessity part of the theorem.

Sufficiency: Consider a controller $u = F(s)y$, where F is a proper rational function. Furthermore, assume the matrix A is Hurwitz (otherwise, apply a stabilizing prefeedback $u_0 = K_0 x$). A simple calculation shows that the precision of aiming of (3.1) with this control is

$$\hat{\mu}(\Psi,F) = \frac{(\min(a,b))^2}{2\|G_F\|_2^2} \tag{A.21}$$

where

$$G_F(s) = (1 + G_s(s)F(s))^{-1}G_n(s) \quad . \tag{A.22}$$

We will construct a family of controllers $\{F_\alpha(s)\}$ which internally stabilize (3.1) and such that $\hat{\mu}(\Psi, F_\alpha) \to \infty$ as $\alpha \to \infty$. Also, it follows from the theory of optimal control that

$$\inf_{\substack{F \\ \text{stabilizing}}} \|G_F\|_2 \geq \inf_{K \in \mathrm{K}} \|G_K\|_2$$

where

$$G_K(s) = D(sI - A - B\bar{K})^{-1}C \quad .$$

Thus, since $\lim_{\alpha \to \infty} \|G_{F_\alpha}\|_2 = 0$ we have $\inf_{K \in \mathrm{K}} \|G_K\|_2 = 0$, i.e., (3.1) is y-srt controllable.

A controller $F(s)$ internally stabilizes (3.1) if $G_F(s)$ and the noise-to-output transfer function

$$\tilde{G}_F(s) = F(s)(1 + G_s(s)F(s))^{-1}G_n(s) \tag{A.23}$$

are stable. Select a proper controller

$$F_\alpha(s) = G_s^{-1}(s)\left[\frac{\alpha^k}{(s+\alpha)^k - \alpha^k}\right] \tag{A.24}$$

where $k \geq \deg[d(s)] - \deg[n(s)]$ and $\alpha > 0$. With this control (A.22) and (A.23) become

$$\begin{aligned} G_{F_\alpha}(s) &= G_\alpha(s) = G_n(s)\,\frac{(s+\alpha)^k - \alpha^k}{(s+\alpha)^k} \quad , \\ \tilde{G}_{F_\alpha}(s) &= \tilde{G}_\alpha(s) = G_n(s)G_s^{-1}(s)\,\frac{\alpha^k}{(s+\alpha)^k} \quad . \end{aligned} \tag{A.25}$$

By assumption $G_n(s)$ is stable, thus $G_\alpha(s)$ is stable. Furthermore, since $G_n(s)$ has zeros at all the rhp zeroes of $G_s(s)$ it follows that $\tilde{G}_\alpha(s)$ is stable. Thus, $F_\alpha(s)$ is a

stabilizing controller for (3.1).

Next note that

$$|G_\alpha(j\omega)|^2 \le |G_n(j\omega)|^2 \left|1 - \left(\frac{\alpha}{j\omega+\alpha}\right)^k\right|^2$$

$$\le 2|G_n(j\omega)|^2$$

and

$$\|G_n\|_2 < \infty \ .$$

Therefore, by the dominated convergence theorem [13]

$$\lim_{\alpha\to\infty} \|G_\alpha\|_2^2 = \|\lim_{\alpha\to\infty} G_\alpha\|_2^2 = 0 \ ,$$

i.e., $\hat{\mu}(\Psi, F_\alpha)\to\infty$ as $\alpha\to\infty$.

Q.E.D.

Proof of Theorem 3.4: We will show that $G_\gamma(s)$ given by (A.16) converges to $G_0(s)$ as $\gamma\to 0$.

From (A.17) and (A.18) we have

$$G_\gamma(s) = \frac{a(s)d_\gamma(s) + n(s)m_\gamma(s)}{d(s)d_\gamma(s)} \ .$$

Also, we know that $d_\gamma(s)$ is the denominator polynomial of $G_\gamma(s)$, thus $d(s)$ divides $a(s)d_\gamma(s) + n(s)m_\gamma(s)$ for all $\gamma > 0$. From the proof of Theorem 3.3 we have

$$G_0(s) = \frac{a(s)n_r(-s) + n_r(s)m_0(s)}{d(s)n_r(-s)}$$

Thus, $a(s)n_r(-s) + n_r(s)m_0(s) = q(s)d(s)$ where $q(s)$ is some polynomial. Furthermore, since the degrees of $a(s)$ and $m_0(s)$ are less than n and $n_r(s)$ has degree l it follows that $q(s)$ has degree less than l.

Note that $G_0(s)$ satisfies, at each nonminimum phase zero of $G_s(s)$ of multiplicity m, the constraints

$$\frac{d^k}{ds^k} G_0(s) \Big|_{s=z} = \frac{d^k}{ds^k} G_n(s) \Big|_{s=z} . \tag{A.26}$$

Therefore,

$$G_0(s) = \frac{q(s)d(s)}{d(s)n_r(-s)} = \frac{q(s)}{n_r(-s)} \tag{A.27}$$

and $q(s)$ is uniquely determined by (A.26)

Q.E.D.

Proof of Theorem 3.5: By assumption, the nonminimum phase zeroes of $G_s(s)$ are distinct. Therefore, we can rewrite $G_0(s)$ as

$$G_0(s) = \sum_{j=1}^{l} \frac{t_j}{\bar{z}_j + s} \tag{A.28}$$

where t_j, $j = 1, \ldots, l$ are some constants. At each z_i we have (from (3.7))

$$G_0(z_i) = \sum_{j=1}^{l} \frac{t_j}{z_i + \bar{z}_j} = G_n(z_i) = g_i \tag{A.29}$$

Thus, (A.29) gives l equations which can be written in matrix notation as

$$Zt = g \quad , \tag{A.30}$$

where $t^T = (t_1, \ldots, t_l)$ and Z and g are defined as in Section 3.

Next we calculate $\|G_0\|_2^2$,

$$\|G_0\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G_0(j\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^{l} \frac{t_j}{\bar{z}_j + j\omega} \sum_{i=1}^{l} \frac{\bar{t}_i}{z_i - j\omega} \, d\omega$$

$$= \sum_{i,j=1}^{l} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{t_j \overline{t_i}}{(\overline{z}_j + j\omega)(z_i - j\omega)} \, d\omega$$

$$= \sum_{i,j=1}^{l} \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{t_j \overline{t_i}}{(\overline{z}_j + s)(z_i - s)} \, ds \qquad \text{(A.31)}$$

Using the calculus of residues to evaluate the integrals appearing in (A.31) gives

$$\int_{-j\infty}^{j\infty} \frac{t_j \overline{t_i}}{(\overline{z}_j + s)(z_i - s)} \, ds = 2\pi j \, \frac{t_j \, \overline{t_i}}{z_i + \overline{z}_j} \qquad \text{(A.32)}$$

Thus,

$$\| G_0 \|_2^2 = \sum_{i,j=1}^{l} \frac{\overline{t_i} t_j}{z_i + \overline{z}_j} = t^H Z t \ . \qquad \text{(A.33)}$$

Substituting, $t = Z^{-1} g$ from (A.30) (note that Z is an invertible Hermitian matrix) gives

$$\| G_0 \|_2^2 = g^H \, Z^{-1} g \qquad \text{(A.34)}$$

and (3.8) follows from (A.34) and (3.5).

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